

A Tight Bound on the Performance of a Minimal-Delay Joint Source-Channel Coding Scheme

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Abstract—An analog source is to be transmitted across a Gaussian channel in more than one channel use per source symbol. This paper derives a lower bound on the asymptotic mean squared error for a strategy that consists of repeatedly quantizing the source, transmitting the quantizer outputs in the first channel uses, and sending the remaining quantization error uncoded in the last channel use. The bound coincides with the performance achieved by a suboptimal decoder studied by the authors in a previous paper, thereby establishing that the bound is tight.

I. INTRODUCTION

This paper gives performance limits of a certain class of encoders for the transmission of a discrete-time memoryless analog source across a discrete-time memoryless Gaussian channel, where the channel can be used n times for each source symbol. The parameter n is arbitrary but fixed, given as part of the problem statement.

It is well known that if the channel noise has variance σ_Z^2 then the average transmit power P and the average mean-squared error D of any communication scheme for this scenario are related by

$$R(D) \leq nC(P), \quad (1)$$

where $R(D)$ is the rate-distortion function of the source under squared-error distortion and $C(P) = 0.5 \log(1 + P/\sigma_Z^2)$ is the cost-constrained capacity of the channel (see *e.g.* [1]). If the source has finite differential entropy $h(S)$ then the rate-distortion function satisfies

$$R(D) \geq h(S) - 0.5 \log(2\pi e D).$$

Applying this bound to (1) and inserting the capacity formula yields

$$D \geq \frac{2^{2h(S)}}{2\pi e} (1 + P/\sigma_Z^2)^{-n} \quad (2)$$

or $D \geq c(1 + \text{SNR})^{-n}$, where we have defined $\text{SNR} = P/\sigma_Z^2$ and $c = 2^{2h(S)}/2\pi e$. As $\text{SNR} \rightarrow \infty$, the squared error distortion scales thus at best as SNR^{-n} .

In this paper we study a communication scheme for this scenario that is extremely simple to implement and has minimal delay, in the sense that it encodes and transmits a single source symbol at a time. It works by quantizing the source and then repeatedly quantizing the quantization error; the quantized

points are sent across the first $n - 1$ channel uses and the last quantization error is sent uncoded in the n^{th} channel use.

We show that no matter how the quantization resolution is chosen (as a function of the SNR) and regardless of the decoder used, the mean squared error achieved by this scheme cannot decay faster than $\text{SNR}^{-n}(\log \text{SNR})^{n-1}$. This asymptotic lower bound coincides with the achievable performance of a suboptimal decoder that we have studied in a previous paper [2]; it is therefore tight.

Transmission schemes of the kind proposed here have been considered before. Indeed, one of the first schemes to transmit an analog source across two uses of a Gaussian channel was suggested by Shannon [3]. Generalizing Shannon's ideas, Wozencraft and Jacobs [4] provided the foundations to analyze source-channel mappings as curves in n -dimensional space. Ziv [5] found important theoretical limitations of such mappings.

Much of the later work is due to Ramstad and his coauthors (see *e.g.* [6], [7], [8]). A proof that the performance of minimal-delay codes is strictly smaller than that of codes with unrestricted delay when $n > 1$ was given in 2008 by Ingber et al. [9].

For $n = 2$, the presented scheme is almost identical to the HSQLC scheme by Coward [10], which uses a numerically optimized quantizer, transmitter and receiver to minimize the mean-squared error (MSE) for finite values of the SNR. Coward conjectured that the right strategy for $n > 2$ would be to repeatedly quantize the quantization error from the previous step, which is exactly what we do here.

Another closely related communication scheme is the *shift-map* scheme due to Chen and Wornell [11]. Vaishampayan and Costa [12] showed in their analysis that it achieves a squared error that scales as $\text{SNR}^{-n+\epsilon}$ for any fixed $\epsilon > 0$ if the relevant parameters are chosen correctly as a function of the SNR. Up to rotation and a different constellation shaping, the shift-map scheme is in fact virtually identical to the one used here, a fact that was pointed out recently by Taherzadeh and Khandani [13]. This suggests that its performance is also limited by the bound derived here.

Other hybrid schemes, such as the one by Shamai et al. [14] or that of Mittal and Phamdo [15], use long block codes for the digital phase and are therefore not directly comparable with

minimum delay schemes.

The rest of this paper is organized as follows. Section II describes our transmission strategy, in Section III we quote the achievability result from our previous paper, and finally Section IV contains our derivation of the mean squared error lower bound.

II. PROPOSED COMMUNICATION SCHEME

To encode a single source letter S into n channel input symbols X_1, \dots, X_n we proceed as follows. Define $E_0 = S$ and recursively compute the pairs (Q_i, E_i) as

$$\begin{aligned} Q_i &= \frac{1}{\beta} \text{int}(\beta E_{i-1}) \\ E_i &= \beta(E_{i-1} - Q_i) \end{aligned} \quad (3)$$

for $i = 1, \dots, n-1$ where $\text{int}(x)$ is the unique integer i such that

$$x \in \left[i - \frac{1}{2}, i + \frac{1}{2} \right)$$

and β is a scaling factor that *grows with the SNR* in a way to be determined later. Q_i is thus a quantized version of the previous round's quantization error, and E_i is the new quantization error scaled up to lie in $[-1/2, 1/2)$. Note that the map $S \mapsto (Q_1, \dots, Q_{n-1}, E_{n-1})$ is one-to-one with the inverse given by

$$S = \sum_{i=1}^{n-1} \frac{1}{\beta^{i-1}} Q_i + \frac{1}{\beta^{n-1}} E_{n-1}. \quad (4)$$

We determine the channel input symbols X_i from the Q_i and from E_{n-1} according to

$$\begin{aligned} X_i &= \sqrt{\frac{P}{\sigma_S^2 + \delta}} Q_i \quad \text{for } i = 1, \dots, n-1 \text{ and} \\ X_n &= \sqrt{\frac{P}{\sigma_E^2}} E_{n-1}, \end{aligned} \quad (5)$$

where $\sigma_E^2 = \text{Var}(E_{n-1})$, and where $\delta > 0$ is some small number. As shown in [2], this ensures that $\mathbb{E}[X_i^2] \leq P$ for all i and for $\beta > \beta_0$ (where β_0 depends on δ). Since we are interested in the large SNR regime and since we have defined β to grow with the SNR, we can assume for the remainder that the power constraint is satisfied.

III. ACHIEVABLE PERFORMANCE

In this section we quote the relevant results of our earlier paper [2], which imply that a suboptimal decoder achieves a mean squared error that scales at least as $\text{SNR}^{-n}(\log \text{SNR})^{n-1}$. While we only considered Gaussian sources in that paper, we actually never used the distribution of the source when we derived the bounds there, so they hold for general continuous sources of bounded variance.

A. Suboptimal Decoder

The encoder outputs X_i are transmitted across the channel, producing at the channel output the symbols

$$Y_i = X_i + Z_i, \quad i = 1, \dots, n,$$

where the Z_i are iid Gaussian random variables of variance σ_Z^2 . To estimate S from Y_1, \dots, Y_n , the decoder first computes separate estimates $\hat{Q}_1, \dots, \hat{Q}_{n-1}$ and \hat{E}_{n-1} , and then combines them to obtain the final estimate \hat{S} .

To estimate the Q_i we use a maximum likelihood (ML) decoder, which yields the minimum distance estimate

$$\hat{Q}_i = \frac{1}{\beta} \arg \min_{j \in \mathbb{Z}} \left| \sqrt{\frac{P}{\sigma_S^2 + \delta}} \frac{j}{\beta} - Y_i \right|. \quad (6)$$

To estimate E_{n-1} , we use a linear minimum mean-square error (LMMSE) estimator (see *e.g.* Scharf [16, Section 8.3]), which computes

$$\hat{E}_{n-1} = \frac{\mathbb{E}[E_{n-1} Y_n]}{\mathbb{E}[Y_n^2]} Y_n. \quad (7)$$

Finally we use (4) to obtain

$$\hat{S} = \sum_{i=1}^{n-1} \frac{1}{\beta^{i-1}} \hat{Q}_i + \frac{1}{\beta^{n-1}} \hat{E}_{n-1}. \quad (8)$$

B. Upper Bounds on the Mean Squared Error

Using the suboptimal decoder described in the previous section, $\mathbb{E}[(S - \hat{S})^2]$ can be broken up into contributions due to the errors in decoding Q_i and E_{n-1} as follows. From (4) and (8), the difference between S and \hat{S} is

$$S - \hat{S} = \sum_{i=1}^{n-1} \frac{1}{\beta^{i-1}} (Q_i - \hat{Q}_i) + \frac{1}{\beta^{n-1}} (E_{n-1} - \hat{E}_{n-1}).$$

The error terms $Q_i - \hat{Q}_i$ depend only on the noise of the respective channel uses and are therefore independent of each other and of $E_{n-1} - \hat{E}_{n-1}$, so we can write the error variance componentwise as

$$\mathbb{E}[(S - \hat{S})^2] = \sum_{i=1}^{n-1} \frac{1}{\beta^{2(i-1)}} \mathcal{E}_{Q,i} + \frac{1}{\beta^{2(n-1)}} \mathcal{E}_E, \quad (9)$$

where $\mathcal{E}_{Q,i} \stackrel{\text{def}}{=} \mathbb{E}[(Q_i - \hat{Q}_i)^2]$ and $\mathcal{E}_E \stackrel{\text{def}}{=} \mathbb{E}[(E_{n-1} - \hat{E}_{n-1})^2]$.

The following two Lemmata, taken directly from [2], give upper bounds on the two types of errors. (The O -notation is defined in Appendix A.)

Lemma 1: For each $i = 1, \dots, n-1$,

$$\mathcal{E}_{Q,i} \in O\left(\exp\{-k \text{SNR}/\beta^2\}\right), \quad (10)$$

where $\text{SNR} = P/\sigma_Z^2$ and $k > 0$ does not depend on SNR.

Lemma 2: The estimation error of E_{n-1} satisfies

$$\mathcal{E}_E/\beta^{2(n-1)} \in O(\text{SNR}^{-1} \beta^{-2(n-1)}). \quad (11)$$

From Lemma 1, β^2 should scale less than linearly in SNR, otherwise the upper bound would be constant. We therefore let

$\beta^2 = \text{SNR}^{1-\epsilon}$, where ϵ is an arbitrary, strictly positive function of SNR. From (10) and (11) we have then

$$\mathcal{E}_{Q,i} \in O(\exp\{-k\text{SNR}^\epsilon\})$$

and

$$\mathcal{E}_E/\beta^{2(n-1)} \in O(\text{SNR}^{-n+(n-1)\epsilon}).$$

Setting $\epsilon(\text{SNR}) = \log((n/k) \log \text{SNR}) / \log \text{SNR}$ we find that the performance achieved by the suboptimal decoder satisfies

$$\mathbb{E}[(\hat{S} - S)^2] \in O(\text{SNR}^{-n}(\log \text{SNR})^{n-1}).$$

The next section shows that this is the best achievable scaling for the given encoder, even if an optimal decoder is used.

IV. DISTORTION LOWER BOUNDS

The goal of this section is to lower bound the scaling of the mean squared error of the transmission strategy described in Section II.

Throughout this section we assume $\beta^2 = \text{SNR}^{1-\epsilon}$, where $\epsilon = \epsilon(\text{SNR})$ is a positive function of SNR. This results in no loss of generality, since for an arbitrary positive function f we can set $\epsilon(\text{SNR}) = 1 - \log(f(\text{SNR})) / \log \text{SNR}$ to get $\beta^2(\text{SNR}) = f(\text{SNR})$.

Note that by (3), the Q_i are completely determined by S . In this section, with a slight abuse of notation, we therefore write $Q_i(s)$ to denote the value of Q_i when $S = s$. We use $E_i(s)$ and $X_i(s)$ in a similar manner. Furthermore, we define $\mathbf{X}(s) = (X_1(s), \dots, X_n(s))$.

The following result, adapted from Ziv [5], is a key ingredient in the proofs of the lemmas that follow.

Lemma 3: Consider a communication system where a continuous-valued source S is encoded into an n -dimensional vector $\mathbf{X}(S)$, sent across n independent parallel AWGN channels with noise variance σ_Z^2 , and decoded at the receiver to produce an estimate \hat{S} . If the density p_S of the source is such that there exists an interval $[A, B]$ and a number $p_{\min} > 0$ such that $p_S(s) \geq p_{\min}$ whenever $s \in [A, B]$, then for any $\Delta \in [0, B - A)$ the mean squared error incurred by the communication system satisfies

$$\mathbb{E}[(\hat{S} - S)^2] \geq p_{\min} \left(\frac{\Delta}{2}\right)^2 \int_A^{B-\Delta} Q(d(s, \Delta)/2\sigma_Z) ds, \quad (12)$$

where $d(s, \Delta) \stackrel{\text{def}}{=} \|\mathbf{X}(s) - \mathbf{X}(s + \Delta)\|$ and

$$Q(x) = \int_x^\infty (1/\sqrt{2\pi}) \exp\{-\xi^2/2\} d\xi.$$

Proof: See Appendix B. ■

The next two lemmata provide two different asymptotic lower bounds on the mean squared error of our transmission strategy, each of which is tighter for a different class of ϵ . They hold regardless of the decoder used. (The Ω -notation is defined in Appendix A.)

Lemma 4: For an arbitrary function $\epsilon(\text{SNR}) \geq 0$, the mean squared error satisfies

$$\mathbb{E}[(\hat{S} - S)^2] \in \Omega(\text{SNR}^{-n+(n-1)\epsilon}).$$

Lemma 5: For an arbitrary function $\epsilon(\text{SNR}) \geq 0$, the mean squared error satisfies

$$\mathbb{E}[(\hat{S} - S)^2] \in \Omega(\text{SNR}^{-1+\epsilon/2} \exp\{-\text{SNR}^\epsilon/k\})$$

where $k > 0$ does not depend on SNR.

Discussion: An immediate consequence of the lemmata is that the theoretically optimal scaling SNR^{-n} is not achievable with the given encoding strategy: by Lemma 4 this would require $\epsilon = 0$, but following Lemma 5 the scaling is at best SNR^{-1} if $\epsilon = 0$. More generally, which one of the two lower bounds decays more slowly and is therefore tighter depends on the scaling of $\epsilon(\text{SNR})$. How to choose $\epsilon(\text{SNR})$ optimally will be the subject of Theorem 7.

Proof of Lemma 4: Assume $\Delta \in [0, \beta^{-(n-1)})$ and define for $j \in \mathbb{Z}$

$$\mathcal{I}_j^\Delta = \left[\left(j - \frac{1}{2}\right)\beta^{-(n-1)}, \left(j + \frac{1}{2}\right)\beta^{-(n-1)} - \Delta \right).$$

It can be verified from (3) that if $s \in \mathcal{I}_j^\Delta$ for some j , the following properties hold: 1) $Q_i(s) = Q_i(s + \Delta)$ for $i = 1, \dots, n-1$, and 2) $E_{n-1}(s + \Delta) - E_{n-1}(s) = \beta^{n-1}\Delta$. From (5) it follows that $s \in \mathcal{I}_j^\Delta$ implies $d(s, \Delta) = \sqrt{P/\sigma_E^2}\beta^{n-1}\Delta$.

We now apply Lemma 3 and restrict the integral to the set $\psi(\Delta) \stackrel{\text{def}}{=} [A, B - \Delta) \cap \bigcup_{j \in \mathbb{Z}} \mathcal{I}_j^\Delta$. The lower bound is then relaxed to give

$$\mathbb{E}[(\hat{S} - S)^2] \geq \frac{p_{\min}}{4} \Delta^2 Q(\sqrt{\text{SNR}/\sigma_E^2}\beta^{n-1}\Delta/2) \int_{\psi(\Delta)} ds.$$

Letting $\Delta = 1/(\sqrt{\text{SNR}}\beta^{n-1})$ and $\beta^2 = \text{SNR}^{1-\epsilon}$ yields

$$\mathbb{E}[(\hat{S} - S)^2] \geq \frac{p_{\min}}{4} \text{SNR}^{-n+(n-1)\epsilon} Q\left(\frac{1}{2\sigma_E}\right) \int_{\psi(\Delta)} ds.$$

The proof is almost complete, but we still have to show that $\int_{\psi(\Delta)} ds$ can be bounded below by a constant for large SNR. The length of a single interval \mathcal{I}_j^Δ is $\beta^{-(n-1)} - \Delta$. Within $[A, B - \Delta)$ there are $(B - A - \Delta)\beta^{n-1}$ such intervals. The total length of all intervals \mathcal{I}_j^Δ in $[A, B - \Delta)$ is therefore

$$\int_{\psi(\Delta)} ds = (B - A - \Delta)(1 - \beta^{n-1}\Delta),$$

which, for the given values of β and Δ , converges to $B - A$ for $\text{SNR} \rightarrow \infty$ and thus can be lower bounded by a constant for SNR greater than some SNR_0 . With this, the proof is complete. ■

Proof of Lemma 5: Observe first that (3) implies $Q_1(s + \beta^{-1}) = Q_1(s) + \beta^{-1}$ and $E_1(s + \beta^{-1}) = E_1(s)$. Since all Q_i and E_i for $i \geq 2$ are by recursion a function of E_1 only, $Q_i(s) = Q_i(s + \beta^{-1})$ for $i = 2, \dots, n-1$, and $E_{n-1}(s) = E_{n-1}(s + \beta^{-1})$. Consequently, $X_i(s) = X_i(s + \beta^{-1})$ for all $i = 2, \dots, n$. By (5) and the above, the Euclidean distance between $\mathbf{X}(s)$ and $\mathbf{X}(s + \beta^{-1})$ is therefore

$$\sqrt{\frac{P}{\sigma_S^2 + \delta}} |Q_1(s) - Q_1(s + \beta^{-1})| = \sqrt{\frac{P}{\sigma_S^2 + \delta}} \beta^{-1}. \quad (13)$$

We now apply Lemma 3 with $\Delta = \beta^{-1}$. The parameter β will be chosen to increase with SNR, therefore $\Delta \in [0, B - A)$ holds for sufficiently large SNR.

Using (13), the resulting bound on the mean squared error is

$$\mathbb{E}[(\hat{S} - S)^2] \geq \frac{p_{\min}}{4} \beta^{-2} Q \left(\sqrt{\frac{\text{SNR}}{\sigma_S^2 + \delta}} \frac{\beta^{-1}}{2} \right) (B - A - \beta^{-1}).$$

Replacing $\beta^2 = \text{SNR}^{1-\epsilon}$ and using the fact that $Q(x)$ converges to $\exp\{-x^2/2\}/\sqrt{2\pi}x$ for $x \rightarrow \infty$ (cf. [17]) we obtain

$$\mathbb{E}[(\hat{S} - S)^2] \geq c \text{SNR}^{-1+\epsilon/2} \exp\{-\text{SNR}^\epsilon/k\}$$

for sufficiently large SNR, with c and k positive constants that do not depend on SNR, thus proving the lemma. ■

The following lemma will be used to prove Theorem 7, the main result of this paper.

Lemma 6: Define $W(x)$ to be the function that satisfies $W(x)e^{W(x)} = x$ for $x > 0$. This function is well defined and is sometimes called the *Lambert W-function* [18]. Then for $\text{SNR} > 1$ and arbitrary real constants $a, b > 0$, and $k > 0$,

$$\text{SNR}^{a+b\epsilon} = \exp\{-\text{SNR}^\epsilon/k\}, \quad (14)$$

if and only if

$$\text{SNR}^\epsilon = bkW(\text{SNR}^{-a/b}/bk). \quad (15)$$

Proof: Let $\text{SNR} > 1$. Since $\text{SNR}^{a+b\epsilon}$ is strictly increasing and $\exp\{-\text{SNR}^\epsilon/k\}$ is strictly decreasing in ϵ , there is at most one solution to (14). Assume now SNR^ϵ is as in (15). Then

$$\exp\{-\text{SNR}^\epsilon/k\} = \exp\{-bW(\text{SNR}^{-a/b}/bk)\}.$$

On the other hand,

$$\begin{aligned} \text{SNR}^{a+b\epsilon} &= \text{SNR}^a \left(bkW(\text{SNR}^{-a/b}/bk) \right)^b \\ &= \left(W(\text{SNR}^{-a/b}/bk) / (\text{SNR}^{-a/b}/bk) \right)^b. \end{aligned}$$

By definition, $W(x)/x = e^{-W(x)}$, so the above is equal to

$$\text{SNR}^{a+b\epsilon} = \exp\{-bW(\text{SNR}^{-a/b}/bk)\},$$

which proves the claim. ■

The following is the main result of the paper.

Theorem 7: For any parameter β and for any decoder, the mean squared error of the transmission strategy described in Section II satisfies

$$\mathbb{E}[(\hat{S} - S)^2] \in \Omega(\text{SNR}^{-n}(\log \text{SNR})^{n-1}).$$

Discussion: The asymptotic lower bound on the mean squared error given by the theorem coincides with the asymptotic performance achieved by the suboptimal decoder in Section III; the bound is therefore asymptotically tight.

Proof of Theorem 7: Define $l_1(\text{SNR}, \epsilon) = \text{SNR}^{-n+(n-1)\epsilon}$ and $l_2(\text{SNR}, \epsilon) = \text{SNR}^{-1+\epsilon/2} \exp\{-\text{SNR}^\epsilon/k\}$. By Lemmata 4 and 5,

$$\mathbb{E}[(\hat{S} - S)^2] \in \Omega(\max(l_1(\text{SNR}, \epsilon), l_2(\text{SNR}, \epsilon))).$$

The optimal parameter $\epsilon(\text{SNR})$ is therefore such that for any SNR

$$\max(l_1(\text{SNR}, \epsilon), l_2(\text{SNR}, \epsilon)) \quad (16)$$

is minimized. Now for any fixed SNR, $l_1(\text{SNR}, \epsilon)$ is increasing in ϵ , and $l_2(\text{SNR}, \epsilon)$ is increasing in ϵ for $0 \leq \epsilon < \xi = \log(k/2)/\log \text{SNR}$ and decreasing in ϵ for $\epsilon \geq \xi$. The maximum in (16) is therefore minimized either for $\epsilon = 0$ or for $\epsilon \geq \xi$ such that $l_1(\epsilon) = l_2(\epsilon)$. As we have remarked earlier, $\epsilon = 0$ leads to a worse performance than that achieved in Section III, and so this cannot be the optimal parameter. We therefore have to choose $\epsilon(\text{SNR})$ such that $l_1(\text{SNR}, \epsilon) = l_2(\text{SNR}, \epsilon)$. Inserting the definitions of l_1 and l_2 and rearranging the terms yields

$$\text{SNR}^{-(n-1)+(n-3/2)\epsilon} = \exp\{-\text{SNR}^\epsilon/k\},$$

which is of the form (14) with $a = -(n-1)$ and $b = n-3/2$. By Lemma 6, for $\text{SNR} > 1$,

$$\text{SNR}^\epsilon = (n-3/2)kW(\text{SNR}^{\frac{2(n-1)}{2n-3}}/((n-3/2)k)).$$

We now use the fact that $W(x)/\log x$ converges to 1 for $x \rightarrow \infty$; this can be shown using L'Hôpital's rule and because the derivative of $W(x)$ is $W(x)/[x(1+W(x))]$ (cf. [18]).

For sufficiently large SNR, therefore, there exists a constant $c > 0$ such that

$$\text{SNR}^\epsilon \geq c(n-3/2)k \left[\frac{2(n-1)}{2n-3} \log \text{SNR} - \log((n-3/2)k) \right],$$

and so $\text{SNR}^\epsilon \in \Omega(\log \text{SNR})$. Plugging this into the bound of Lemma 4 we finally obtain¹

$$\mathbb{E}[(\hat{S} - S)^2] \in \Omega(\text{SNR}^{-n}(\log \text{SNR})^{n-1}),$$

and no choice of $\epsilon(\text{SNR})$ can improve this bound. ■

V. CONCLUSIONS

We have analyzed the source-channel coding strategy of repeatedly quantizing an analog source and transmitting the quantizer outputs and the remaining quantization error uncoded across n Gaussian channels. We have shown that if the quantization resolution of the encoder is chosen optimally and if the optimal decoder is used then the mean squared error scales at best as $\text{SNR}^{-n}(\log \text{SNR})^{n-1}$. Furthermore, as our previous paper showed, a simple suboptimal decoder is sufficient to achieve this scaling, so the bound is tight.

The question whether any minimal delay scheme can asymptotically perform better than $\text{SNR}^{-n}(\log \text{SNR})^{n-1}$ is still open at this time.

APPENDIX A ASYMPTOTIC NOTATION

The “ O ” and “ Ω ” asymptotic notation used at various points in the paper is defined as follows. Let $f(x)$ and $g(x)$ be two functions defined on \mathbb{R} . We write

$$f(x) \in O(g(x))$$

¹If $a(x) \in \Omega(f(x))$ and $b(x) \in \Omega(g(x))$, then $a(x)b(x)^m \in \Omega(f(x)g(x)^m)$.

if and only if there exists an x_0 and a constant c such that

$$f(x) \leq cg(x)$$

for all $x > x_0$.

Similarly, we write $f(x) \in \Omega(x)$ if \leq is replaced by \geq in the above definition.

APPENDIX B PROOF OF ZIV'S LOWER BOUND (LEMMA 3)

If we condition the mean squared error on S and use the assumption on p_S we obtain

$$\mathbb{E}[(\hat{S} - S)^2] \geq p_{\min} \int_A^B \mathbb{E}[(\hat{S} - S)^2 | s] ds.$$

For $\Delta \in [0, B - A]$ we can further bound this in two ways:

$$\begin{aligned} \mathbb{E}[(\hat{S} - S)^2] &\geq p_{\min} \int_A^{B-\Delta} \mathbb{E}[(\hat{S} - S)^2 | s] ds \\ \mathbb{E}[(\hat{S} - S)^2] &\geq p_{\min} \int_{A+\Delta}^B \mathbb{E}[(\hat{S} - S)^2 | s] ds \\ &= p_{\min} \int_A^{B-\Delta} \mathbb{E}[(\hat{S} - S - \Delta)^2 | s + \Delta] ds. \end{aligned}$$

Averaging the two lower bounds yields

$$\mathbb{E}[(\hat{S} - S)^2] \geq \frac{p_{\min}}{2} \int_A^{B-\Delta} \left(\mathbb{E}[(\hat{S} - S)^2 | s] + \mathbb{E}[(\hat{S} - S - \Delta)^2 | s + \Delta] \right) ds, \quad (17)$$

and applying Markov's inequality to the expectation terms leads to

$$\mathbb{E}[(\hat{S} - S)^2 | s] \geq \left(\frac{\Delta}{2} \right)^2 \Pr[|\hat{S} - S| \geq \Delta/2 | s] \quad (18)$$

and

$$\begin{aligned} \mathbb{E}[(\hat{S} - S - \Delta)^2 | s + \Delta] \\ \geq \left(\frac{\Delta}{2} \right)^2 \Pr[|\hat{S} - S - \Delta| \geq \Delta/2 | s + \Delta]. \end{aligned} \quad (19)$$

Now suppose that we use the communication system in question for binary signaling. We want to send either s or $s + \Delta$; at the decoder we use the estimate \hat{S} to decide for s or $s + \Delta$ depending on which one \hat{S} is closer to. When s is sent, the decoder makes an error only if $|\hat{S} - s| \geq \Delta/2$; when $s + \Delta$ is sent, it makes an error only if $|\hat{S} - s - \Delta| \geq \Delta/2$. The conditional error probabilities therefore satisfy $\Pr[\text{error} | s] \leq \Pr[|\hat{S} - S| \geq \Delta/2 | s]$ and

$\Pr[\text{error} | s + \Delta] \leq \Pr[|\hat{S} - S - \Delta| \geq \Delta/2 | s + \Delta]$. Applying this to (18) and (19) and inserting the result in (17) yields

$$\mathbb{E}[(\hat{S} - S)^2] \geq p_{\min} \left(\frac{\Delta}{2} \right)^2 \int_A^{B-\Delta} P_e(s, \Delta) ds, \quad (20)$$

where $P_e(s, \Delta) = (\Pr[\text{error} | s] + \Pr[\text{error} | s + \Delta]) / 2$ is the average error probability.

If s and $s + \Delta$ are picked with equal probability and transmitted across n parallel Gaussian channels as $\mathbf{X}(s)$ and $\mathbf{X}(s + \Delta)$, and if $d(s, \Delta) = \|\mathbf{X}(s) - \mathbf{X}(s + \Delta)\|$, then the error probability of the MAP decoder is $Q(d(s, \Delta)/2\sigma_Z)$, a standard result of communication theory (see e.g. [4, Section 4.5]). Because the MAP decoder minimizes the error probability, $Q(d(s, \Delta)/2\sigma_Z) \leq P_e(s, \Delta)$, which, when inserted into (20), completes the proof. ■

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